

Some conjectures on addition and multiplication of complex (real) numbers

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Abstract. We discuss conjectures related to the following two conjectures:**(I)** (see [7]) for each complex numbers x_1, \dots, x_n there exist rationals $y_1, \dots, y_n \in [-2^{n-1}, 2^{n-1}]$ such that

$$\forall i \in \{1, \dots, n\} \ (x_i = 1 \Rightarrow y_i = 1) \quad (1)$$

$$\forall i, j, k \in \{1, \dots, n\} \ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k) \quad (2)$$

(II) (see [6], [7]) for each complex (real) numbers x_1, \dots, x_n there exist complex (real) numbers y_1, \dots, y_n such that

$$\forall i \in \{1, \dots, n\} \ |y_i| \leq 2^{2^{n-2}} \quad (3)$$

$$\forall i \in \{1, \dots, n\} \ (x_i = 1 \Rightarrow y_i = 1) \quad (4)$$

$$\forall i, j, k \in \{1, \dots, n\} \ (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k) \quad (5)$$

$$\forall i, j, k \in \{1, \dots, n\} \ (x_i \cdot x_j = x_k \Rightarrow y_i \cdot y_j = y_k) \quad (6)$$

Mathematics Subject Classification: 03B30, 12D99, 14P05, 15A06, 15A09**Keywords:** system of linear equations, system of polynomial equations, solution with minimal Euclidean norm, least-squares solution with minimal Euclidean normFor a positive integer n we define the set of equations W_n by

$$W_n = \{x_i = 1 : 1 \leq i \leq n\} \cup \{x_i + x_j = x_k : 1 \leq i \leq j \leq n, 1 \leq k \leq n\}$$

Let $S \subseteq W_n$ be a system consistent over \mathbb{C} . Then S has a solution which consists of rationals belonging to $[-(\sqrt{5})^{n-1}, (\sqrt{5})^{n-1}]$, see [7, Theorem 9]. Conjecture **(I)** states that S has a solution which consists of rationals belonging to $[-2^{n-1}, 2^{n-1}]$.Concerning Conjecture **(I)**, estimation by 2^{n-1} is the best estimation. Indeed, the system

$$\left\{ \begin{array}{rcl} x_1 & = & 1 \\ x_1 + x_1 & = & x_2 \\ x_2 + x_2 & = & x_3 \\ x_3 + x_3 & = & x_4 \\ & \dots & \\ x_{n-1} + x_{n-1} & = & x_n \end{array} \right.$$

has precisely one complex solution: $(1, 2, 4, 8, \dots, 2^{n-2}, 2^{n-1})$. Concerning Conjecture **(II)**, for $n = 1$ estimation by $2^{2^{n-2}}$ can be replaced by estimation by 1. For $n > 1$ estimation by $2^{2^{n-2}}$ is the best estimation, see [7].

For each consistent system $S \subseteq W_n$ there exists $J \subseteq \{1, \dots, n\}$ such that the system $S \cup \{x_i + x_i = x_i : i \in J\}$ has a unique solution (x_1, \dots, x_n) , see the proof of Theorem 9 in [7]. If any $S \subseteq W_n$ has a unique solution (x_1, \dots, x_n) , then by Cramer's rule each x_i is a quotient of two determinants. Since these determinants have entries among $-1, 0, 1, 2$, each x_i is rational.

For proving Conjecture **(I)**, without loss of generality we can assume that the equation $x_1 = 1$ belongs to S and all equations $x_i = 1$ ($i > 1$) do not belong to S . Indeed, if $i > 1$ and the equation $x_i = 1$ belongs to S , then we replace x_i by x_1 in all equations belonging to S . In this way the problem reduces to the same problem with a smaller number of variables, for details see the text after Conjecture 3. Therefore, for proving Conjecture **(I)** it is sufficient to consider only these systems $S \subseteq W_n$ of n equations which satisfy the following conditions:

S contains the equation $x_1 = 1$ and $n - 1$ equations of the form $x_i + x_j = x_k$ ($i, j, k \in \{1, \dots, n\}$),

joining the equation $x_1 = 0$ and the aforementioned $n - 1$ equations of the form $x_i + x_j = x_k$, we get a system of n linearly independent equations.

By the observations from the last two paragraphs, the following code in *MuPAD* yields a probabilistic confirmation of Conjecture **(I)**. The value of n is set, for example, to 5. The number of iterations is set, for example, to 1000. We use the algorithm which terminates with probability 1. For another algorithm, implemented in *Mathematica*, see [4].

```
SEED:=time():
r:=random(1..5):
idmatrix:=matrix::identity(5):
u:=linalg::row(idmatrix,i) $i=1..5:
max_norm:=1:
for k from 1 to 1000 do
a:=linalg::row(idmatrix,1):
rank:=1:
while rank<5 do
m:=matrix(u[r()])+matrix(u[r()])-matrix(u[r()]):
```

```

a1:=linalg::stackMatrix(a,m):
rank1:=linalg::rank(a1):
if rank1 > rank then a:=linalg::stackMatrix(a,m) end_if:
rank:=linalg::rank(a):
end_while:
x:=(a^-1)*linalg::col(idmatrix,1):
max_norm:=max(max_norm,norm(x)):
print(max_norm):
end_for:

```

Conjecture (I) holds true for each $n \leq 4$. It follows from the following Observation 1.

Observation 1 ([7, p. 23]). If $n \leq 4$ and $(x_1, \dots, x_n) \in \mathbb{C}^n$ solves S , then some $(\widehat{x}_1, \dots, \widehat{x}_n)$ solves S , where each \widehat{x}_i is suitably chosen from $\{x_i, 0, 1, 2, \frac{1}{2}\} \cap \{r \in \mathbb{Q} : |r| \leq 2^{n-1}\}$.

Let $\mathbf{Ax} = \mathbf{b}$ be the matrix representation of the system S , and let \mathbf{A}^\dagger denote Moore-Penrose pseudoinverse of \mathbf{A} . The system S has a unique solution \mathbf{x}_0 with minimal Euclidean norm, and this element is given by $\mathbf{x}_0 = \mathbf{A}^\dagger \mathbf{b}$, see [5, p. 423].

For any system $S \subseteq W_n$, a vector $\mathbf{x} \in \mathbb{C}^n$ is said to be a least-squares solution if \mathbf{x} minimizes the Euclidean norm of $\mathbf{Ax} - \mathbf{b}$, see [1, p. 104]. It is known that $\mathbf{x}_0 = \mathbf{A}^\dagger \mathbf{b}$ is a unique least-squares solution with minimal Euclidean norm, see [1, p. 109].

Since \mathbf{A} has rational entries (the entries are among $-1, 0, 1, 2$), \mathbf{A}^\dagger has also rational entries, see [2, p. 69] and [3, p. 193]. Since \mathbf{b} has rational entries (the entries are among 0 and 1), $\mathbf{x}_0 = \mathbf{A}^\dagger \mathbf{b}$ consists of rationals.

Conjecture 1. The solution (The least-squares solution) \mathbf{x}_0 consists of numbers belonging to $[-2^{n-1}, 2^{n-1}]$.

Conjecture 1 restricted to the case when $\text{card } S \leq n$ implies Conjecture (I). The following code in *MuPAD* yields a probabilistic confirmation of Conjecture 1 restricted to the case when $\text{card } S \leq n$. The value of n is set, for example, to 5. The number of iterations is set, for example, to 1000.

```

SEED:=time():
r:=random(1..5):
idmatrix:=matrix::identity(5):
u:=linalg::row(idmatrix,i) $i=1..5:
max_norm:=1:
for k from 1 to 1000 do

```

```

b:=[1]:
c:=linalg::row(idmatrix,1):
for w from 1 to 4 do
h:=0:
h1:=r():
h2:=r():
h3:=r():
m1:=matrix(u[h1]):
m2:=matrix(u[h2]):
m3:=matrix(u[h3]):
m:=m1+m2-m3:
c:=linalg::stackMatrix(c,m):
if h3=h2 then h:=1 end_if:
b:=append(b,h):
a:=linalg::pseudoInverse(c):
x:=a*matrix(b):
max_norm:=max(max_norm,norm(x)):
end_for:
print(max_norm):
end_for:

```

The following Conjecture 2 implies Conjecture (I), see [7].

Conjecture 2 ([7]). Let \mathbf{B} be a matrix with $n-1$ rows and n columns, $n \geq 2$. Assume that each row of \mathbf{B} , after deleting all zeros, forms a sequence belonging to

$$\{\langle 1 \rangle, \langle -1, 2 \rangle, \langle 2, -1 \rangle, \langle -1, 1, 1 \rangle, \langle 1, -1, 1 \rangle, \langle 1, 1, -1 \rangle\}$$

We conjecture that after deleting any column of \mathbf{B} we get the matrix whose determinant has absolute value less than or equal to 2^{n-1} .

Conjecture 3. If a system $S \subseteq W_n$ has a unique solution (x_1, \dots, x_n) , then this solution consists of rationals whose nominators and denominators belong to $[-2^{n-1}, 2^{n-1}]$.

Conjecture 3 implies Conjecture (I). The *MuPAD* code below confirms Conjecture 3 probabilistically. As previously, the value of n is set to 5, the number of iterations is set to 1000. We declare that

$$\{i \in \{1, 2, 3, 4, 5\} : \text{the equation } x_i = 1 \text{ belongs to } S\} = \{1\},$$

but this does not decrease the generality. Indeed, $(0, \dots, 0)$ solves S if all equations $x_i = 1$ do not belong to S . In other cases, let

$$I = \{i \in \{1, \dots, n\} : \text{the equation } x_i = 1 \text{ belongs to } S\},$$

and let $i = \min(I)$. For each $j \in I$ we replace x_j by x_i in all equations belonging to S . We obtain an equivalent system \widehat{S} with $n - \text{card}(I) + 1$ variables. The system \widehat{S} has a unique solution $(t_1, \dots, t_{n-\text{card}(I)+1})$, and the equation $x_j = 1$ belongs to \widehat{S} if and only if $j = i$. By permuting variables, we may assume that $i = 1$.

```
SEED:=time():
r:=random(1..5):
idmatrix:=matrix::identity(5):
u:=linalg::row(idmatrix,i) $i=1..5:
abs_numer_denom:=[1]:
for k from 1 to 1000 do
c:=linalg::row(idmatrix,1):
rank:=1:
while rank<5 do
m:=matrix(u[r()])+matrix(u[r()])-matrix(u[r()]):
if linalg::rank(linalg::stackMatrix(c,m))>rank
then c:=linalg::stackMatrix(c,m) end_if:
rank:=linalg::rank(c):
end_while:
a:=(c^-1)*linalg::col(idmatrix,1):
for n from 2 to 5 do
abs_numer_denom:=append(abs_numer_denom,abs( numer(a[n]))):
abs_numer_denom:=append(abs_numer_denom,abs( denom(a[n]))):
end_for:
abs_numer_denom:=listlib::removeDuplicates(abs_numer_denom):
print(max(abs_numer_denom)):
end_for:
```

The *MuPAD* code below completely confirms Conjecture 3 for $n = 5$. We declare that

$$\{i \in \{1, 2, 3, 4, 5\} : \text{the equation } x_i = 1 \text{ belongs to } S\} = \{1\},$$

but this does not decrease the generality.

```
p:=[]:
idmatrix:=matrix::identity(5):
u:=linalg::row(idmatrix,i) $i=1..5:
```

```

for r1 from 1 to 5 do
for r2 from 1 to 5 do
for r3 from 1 to 5 do
m1:=matrix(u[r1]):
m2:=matrix(u[r2]):
m3:=matrix(u[r3]):
m:=m1+m2-m3:
p:=append(p,m):
end_for:
end_for:
end_for:
p:=listlib::removeDuplicates(p):
p:=listlib::setDifference(p,[linalg::row(idmatrix,1)]):
abs_numer_denom:=[]:
s1:=nops(p)-1:
s2:=nops(p)-2:
s3:=nops(p)-3:
for n3 from 1 to s3 do
w2:=n3+1:
for n2 from w2 to s2 do
w1:=n2+1:
for n1 from w1 to s1 do
w0:=n1+1:
for n0 from w0 to nops(p) do
c3:=linalg::stackMatrix(linalg::row(idmatrix,1),p[n3]):
c2:=linalg::stackMatrix(c3,p[n2]):
c1:=linalg::stackMatrix(c2,p[n1]):
c:=linalg::stackMatrix(c1,p[n0]):
if linalg::rank(c)=5 then
a:=(c^-1)*linalg::col(idmatrix,1):
for n from 2 to 5 do
abs_numer_denom:=append(abs_numer_denom,abs(numer(a[n]))):
abs_numer_denom:=append(abs_numer_denom,abs(denom(a[n]))):
end_for:
abs_numer_denom:=listlib::removeDuplicates(abs_numer_denom):

```

```

end_if:
end_for:
end_for:
abs_numer_denom:=sort(abs_numer_denom):
print(abs_numer_denom):
end_for:
end_for:

```

The following Conjecture 4 implies Conjecture (I), because each consistent system $S \subseteq W_n$ can be enlarged to a system $\tilde{S} \subseteq W_n$ with a unique solution (x_1, \dots, x_n) and $(x_1, \dots, x_n) \in \mathbb{Q}^n$.

Conjecture 4. Let rationals x_1, \dots, x_n satisfy $|x_1| \leq |x_2| \leq \dots \leq |x_n|$, and for each $y_1, \dots, y_n \in \mathbb{Q}$, if

$$\forall i \in \{1, \dots, n\} (x_i = 1 \Rightarrow y_i = 1)$$

and

$$\forall i, j, k \in \{1, \dots, n\} (x_i + x_j = x_k \Rightarrow y_i + y_j = y_k),$$

then $(x_1, \dots, x_n) = (y_1, \dots, y_n)$. We conjecture that for each $i \in \{1, \dots, n-1\}$ the inequality $|x_i| \geq 1$ implies $|x_{i+1}| \leq 2 \cdot |x_i|$.

Concerning Conjecture 4, the assumption $|x_i| \geq 1$ is necessary to state that $|x_{i+1}| \leq 2 \cdot |x_i|$. As a trivial counterexample we have $(0, 1)$, $(\frac{1}{4}, \frac{3}{4}, 1, \frac{3}{2}, 2, 3)$ is a counterexample which consists of positive rationals alone. The *MuPAD* code below probabilistically confirms Conjecture 4. The value of n is set, for example, to 5. The number of iterations is set, for example, to 1000.

```

SEED:=time():
r:=random(1..5):
idmatrix:=matrix::identity(5):
u:=linalg::row(idmatrix,i) $i=1..5:
max_ratio:=1:
for k from 1 to 1000 do
a:=linalg::row(idmatrix,1):
rank:=1:
while rank<5 do
m:=matrix(u[r()])+matrix(u[r()])-matrix(u[r()]):
a1:=linalg::stackMatrix(a,m):
rank1:=linalg::rank(a1):

```

```

if rank1 > rank then a:=linalg::stackMatrix(a,m) end_if:
rank:=linalg::rank(a):
end_while:
x:=(a^-1)*linalg::col(idmatrix,1):
xx:=[max(1,abs(x[i])) $i=1..5]:
xxx:=sort(xx):
maxratio:=max([xxx[i+1]/xxx[i] $i=1..4]):
max_ratio:=max(max_ratio,maxratio):
print(max_ratio):
end_for:

```

For a positive integer n we define the set of equations E_n by

$$E_n = \{x_i = 1 : 1 \leq i \leq n\} \cup$$

$$\{x_i + x_j = x_k : 1 \leq i \leq j \leq n, 1 \leq k \leq n\} \cup \{x_i \cdot x_j = x_k : 1 \leq i \leq j \leq n, 1 \leq k \leq n\}$$

Let $T \subseteq E_n$ be a system consistent over \mathbb{C} (over \mathbb{R}). Conjecture **(II)** states that T has a complex (real) solution which consists of numbers whose absolute values belong to $[0, 2^{2^{n-2}}]$. Both for complex and real case, we conjecture that each solution of T with minimal Euclidean norm consists of numbers whose absolute values belong to $[0, 2^{2^{n-2}}]$. This conjecture implies Conjecture **(II)**. Conjecture **(II)** holds true for each $n \leq 4$. It follows from the following Observation 2.

Observation 2 ([7, p. 7]). If $n \leq 4$ and $(x_1, \dots, x_n) \in \mathbb{C}^n$ (\mathbb{R}^n) solves T , then some $(\widehat{x}_1, \dots, \widehat{x}_n)$ solves T , where each \widehat{x}_i is suitably chosen from $\{x_i, 0, 1, 2, \frac{1}{2}\} \cap \{z \in \mathbb{C} (\mathbb{R}) : |z| \leq 2^{2^{n-2}}\}$.

Let us consider the following four conjectures, analogical conjectures seem to be true for \mathbb{R} .

(5a) If a system $S \subseteq E_n$ is consistent over \mathbb{C} and maximal with respect to inclusion, then each solution of S belongs to

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| \leq 2^{2^{n-2}} \wedge \dots \wedge |x_n| \leq 2^{2^{n-2}}\}.$$

(5b) If a system $S \subseteq E_n$ is consistent over \mathbb{C} and maximal with respect to inclusion, then S has a finite number of solutions (x_1, \dots, x_n) .

(5c) If the equation $x_1 = 1$ belongs to $S \subseteq E_n$ and S has a finite number of complex solutions (x_1, \dots, x_n) , then each such solution belongs to

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| \leq 2^{2^{n-2}} \wedge \dots \wedge |x_n| \leq 2^{2^{n-2}}\}.$$

(5d) If a system $S \subseteq E_n$ has a finite number of complex solutions (x_1, \dots, x_n) , then each such solution belongs to

$$\{(x_1, \dots, x_n) \in \mathbb{C}^n : |x_1| \leq 2^{2^{n-1}} \wedge \dots \wedge |x_n| \leq 2^{2^{n-1}}\}.$$

Conjecture 5a strengthens Conjecture (II) for \mathbb{C} . The conjunction of Conjectures 5b and 5c implies Conjecture 5a.

Concerning Conjecture 5d, for $n = 1$ estimation by $2^{2^{n-1}}$ can be replaced by estimation by 1. For $n > 1$ estimation by $2^{2^{n-1}}$ is the best estimation. Indeed, the system

$$\left\{ \begin{array}{lcl} x_1 + x_1 & = & x_2 \\ x_1 \cdot x_1 & = & x_2 \\ x_2 \cdot x_2 & = & x_3 \\ x_3 \cdot x_3 & = & x_4 \\ & \dots & \\ x_{n-1} \cdot x_{n-1} & = & x_n \end{array} \right.$$

has precisely two complex solutions, $(0, \dots, 0)$, $(2, 4, 16, 256, \dots, 2^{2^{n-2}}, 2^{2^{n-1}})$.

The following code in *MuPAD* yields a probabilistic confirmation of Conjectures 5b and 5c. The value of n is set, for example, to 5. The number of iterations is set, for example, to 1000.

```
SEED:=time():
p:=[v-1,x-1,y-1,z-1]:
var:=[1,v,x,y,z]:
for i from 1 to 5 do
for j from i to 5 do
for k from 1 to 5 do
p:=append(p,var[i]+var[j]-var[k]):
p:=append(p,var[i]*var[j]-var[k]):
end_for:
end_for:
end_for:
p:=listlib::removeDuplicates(p):
max_abs_value:=1:
for r from 1 to 1000 do
q:=combinat::permutations::random(p):
syst:=[t-v-x-y-z]:
```

```

w:=1:
repeat
if groebner::dimension(append(syst,q[w]))>-1
then syst:=append(syst,q[w]) end_if:
w:=w+1:
until (groebner::dimension(syst)=0 or w>nops(q)) end:
d:=groebner::dimension(syst):
if d>0 then print("Conjecture 5b is false") end_if:
if d=0 then
sol:=numeric::solve(syst):
for m from 1 to nops(sol) do
for n from 2 to 5 do
max_abs_value:=max(max_abs_value,abs(sol[m][n][2])):
end_for:
end_for:
end_if:
print(max_abs_value);
end_for:

```

If we replace

p:=[v-1,x-1,y-1,z-1]:	by	p:=[]:
var:=[1,v,x,y,z]:	by	var:=[u,v,x,y,z]:
max_abs_value:=1:	by	max_abs_value:=0:
syst:=[t-v-x-y-z]:	by	syst:=[t-u-v-x-y-z]:
for n from 2 to 5 do	by	for n from 2 to 6 do

then we get a code for a probabilistic confirmation of Conjecture 5d.

It seems that for each integers x_1, \dots, x_n there exist integers $y_1, \dots, y_n \in [-2^{n-1}, 2^{n-1}]$ with properties (1) and (2), cf. [7, Theorem 10]. However, not for each integers x_1, \dots, x_n there exist integers y_1, \dots, y_n with properties (3)-(6), see [7, pp. 15–16].

The author used MuPAD Pro 4.0.6. SciFace Software GmbH & Co. KG, the maker of MuPAD Pro, has been acquired by The MathWorks, the maker of MATLAB technical computing software. MuPAD Pro is no longer sold as a standalone product. A new package, Symbolic Math Toolbox 5.1 requires MATLAB and provides large compatibility with existing MuPAD Pro applications.

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